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Characteristic Classes of Surface Bundles with Cross Sections

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Introduction. In our previous papers [M 1,2,3,4], we have defined the notion of characteristic classes of surface bundles, derived several properties of them and in particular we proved that they are highly non-trivial. In this paper we treat the case of surface bundles <u>with cross sections</u> and obtain similar results. More concretely, in §1 we define characteristic classes of surface bundles with cross sections and in §2 we compute them for the examples of surface bundles constructed in [M 1]. In the final section (§3) we prove our main result (Theorem 3-1) which shows that our characteristic classes are highly non-trivial. To prove it we make an essential use of a fundamental result of Harer [H] about stability of the homology of the mapping class groups of surfaces.

Throughout this paper we use the ideas and terminologies of [M 1] freely.

1. Surface bundles with cross sections

Let Σ_g be a closed orientable surface of genus g. In this paper we always assume that $g \geq 2$. Let $\pi: E \to X$ be an oriented Σ_g-bundle and let ξ be its tangent bundle. Assume that there are given cross sections

$$\sigma_i: X \to E \qquad (i=1,\dots,p,p+1,\dots,p+q)$$

such that the images $\sigma_i(X)$ are mutually disjoint. We also assume that the normal bundles of the images of the last q cross sections are trivialized, namely there are given trivializations

$$N(\sigma_i(X)) = X \times D^2 \qquad (i=p+1,\dots,p+q)$$

where $N(\sigma_i(X))$ denotes a tubular neighbourhood of $\sigma_i(X)$. Let $e = e(\xi) \in H^2(E;\mathbb{Z})$ be the Euler class of the surface bundle $\pi\colon E \to X$. As in [M 1], we set

$$e_i = \pi_*(e^{i+1}) \in H^{2i}(X;\mathbb{Z})$$

where $\pi_*\colon H^{2(i+1)}(E;\mathbb{Z}) \to H^{2i}(X;\mathbb{Z})$ is the Gysin homomorphism. Next we set

$$s_i = \sigma_i^*(e) \in H^2(X;\mathbb{Z}) \qquad (i=1,\dots,p).$$

It is clear that the cohomology classes e_i and s_i of the base space X behave naturally with respect to bundle maps of surface bundles with cross sections. We call them characteristic classes of surface bundles with cross sections.

There is one natural way to obtain a surface bundle with a cross section out of a given surface bundle $\pi\colon E \to X$. Namely let

$$\begin{tikzcd} E^* \arrow[r, "q"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"'] & X \end{tikzcd}$$

be the pull back of the bundle $\pi\colon E \to X$ by the map π itself so that $E^* = \{(z,z')\ E \times E;\ \pi(z) = \pi(z')\}$, $\pi'(z,z') = z$ and $q(z,z') = z'$. Then the surface bundle $\pi'\colon E^* \to E$ has a natural cross section $\sigma\colon E \to E^*$ given by $\sigma(z) = (z,z)$. The Euler class of π' is clearly equal to $q^*(e)$. Hence the s-class of it is given by

$$s = \sigma^* q^*(e) = e \quad H^2(E;\mathbb{Z}).$$

This fact will play an important role in the proof of our main result given in §3.

Next we describe the classifying spaces for surface bundles with cross sections. Let $\Sigma_{g,q}$ be a compact orientable surface of genus g with q boundary components and let $x_1,\ldots,x_p$ be p fixed points on $\mathrm{Int}\,\Sigma_{g,q}$. Let $D^p_{g,q}$ be the group of all orientation preserving diffeomorphisms of $\Sigma_{g,q}$ such that they restrict to the identity on the boundary $\partial\Sigma_{g,q}$ and also fix the p points $x_1,\ldots,x_p$. We denote $M^p_{g,q}$ for the group of path components of $D^p_{g,q}$. It is usually called the (pure) mapping class group of genus g with p punctures and q boundary components. Let $E^p_{g,q}$ be the connected component of the identity of $D^p_{g,q}$.

Proposition 1-1. $E^p_{g,q}$ is contractible.

Proof. The cases where $p = 0$ are nothing but the main theorems of Earle-Eells [EE] (q=0) and Earle-Schatz [ES] (q: arbitrary). The general cases follow from them by an inductive argument using the natural sequence

$$D^p_{g,q} \longrightarrow D^{p-1}_{g,q} \longrightarrow \Sigma_{g,q} \setminus \{x_1,\dots,x_{p-1}\}$$

which is a locally trivial fibration.

Corollary 1-2. The classifying space $BD^p_{g,q}$ of the topological group $D^p_{g,q}$ is an Eilenberg-MacLane space $K(M^p_{g,q},1)$.

Now it is easy to see that the space $BD^p_{g,q}$ classifies those oriented Σ_g-bundles which have $p+q$ disjoint cross sections the normal bundles of the last q of which are trivialized. Therefore we can consider elements e_i and s_i as cohomology classes of the space $BD^p_{g,q}$ and hence cohomology classes of the group $M^p_{g,q}$ by Corollary 1-2. Thus we obtain a homomorphism

$$\phi\colon \mathbb{Q}[e_1,\dots,e_{g-2},s_1,\dots,s_p] \longrightarrow H^*(M^p_{g,q};\mathbb{Q}).$$

We know that the above homomorphism has a large kernel (see [M 1,3,4] for informations on Ker ϕ). However we prove in §3 that it is injective up to degree $\frac{1}{3}g$.

2. Computations of characteristic numbers

In this section we compute characteristic numbers of those surface bundles which were constructed in [M 1]. We first recall the main constructions of that paper. Thus let $\pi\colon E \to X$ be an oriented surface bundle and assume that the total space E is an <u>iterated surface bundle</u>, namely a member of the class C defined in [M 1]. Then for any natural number m, we can consider an m-construction on it:

$$\begin{array}{ccccc} \tilde{E}^* & \xrightarrow{r} & E^* & \xrightarrow{q} & E \\ \tilde{\pi}\downarrow & & \downarrow\pi' & & \downarrow\pi \\ \tilde{E} & \xrightarrow[\bar{r}]{} & E & \xrightarrow[\pi]{} & X. \end{array}$$

Here $\pi': E^* \to E$ is the pull back of $\pi: E \to X$ by the map π as was already introduced in §1 and up to pull back maps induced by finite coverings of the base spaces, the map $r: \tilde{E}^* \to E^*$ is essentially an m-fold cyclic ramified covering followed by an m-fold fibre-wise covering (see [M 1] for details). The ramification locus $\tilde{D} \subset \tilde{E}^*$ is a codimension two submanifold of $\tilde{E}^*$ and is the full inverse image under the map r of the "diagonal" $D = \{(z,z) \in E^*;\ z \in E\} \subset E^*$. Let $\nu \in H^2(E^*;\mathbb{Z})$ (resp. $\tilde{\nu} \in H^2(\tilde{E}^*;\mathbb{Z})$) be the Poincaré dual of D (resp. $\tilde{D}$). We write e, e_i (resp. $\tilde{e}$, $\tilde{e}_i$) for the Euler class and the e_i-class of the bundle $\pi: E \to X$ (resp. $\tilde{\pi}: \tilde{E}^* \to \tilde{E}$). The following assertion has been proved in [M 1] (Proposition 5-4).

Proposition 2-1. (i) $\tilde{e} = r^*\{q^*(e) - \frac{m-1}{m}\nu\} \in H^2(\tilde{E}^*;\mathbb{Q})$.
(ii) $\tilde{e}_i = m^2\bar{r}^*\{\pi^*(e_i) - (1-m^{-(i+1)})e^i\} \in H^{2i}(\tilde{E};\mathbb{Q})$.

If $\dim X = 2n$, then for each partition $I = \{i_1,\ldots,i_r\}$ of n, we have the corresponding characteristic number

$$e_I[X] = e_{i_1}\cdots e_{i_r}[X].$$

For each subset $J = \{j_1,\ldots,j_s\}$ of a partition $I = \{i_1,\ldots,i_r\}$ of some natural number, we express the complement $J^c = I\setminus J$ as $J^c = \{k_1,\ldots,k_t\}$ $(s+t=r)$. With these notations we have

Proposition 2-2. Let $\dim E = 2(n+1)$ and let $I = \{i_1,\ldots,i_r\}$ be a partition of $n+1$. Then the I-th characteristic number of the surface bundle $\tilde{\pi}\colon \tilde{E}^* \to \tilde{E}$ is given by

$$\tilde{e}_I[\tilde{E}] = dm^{2r} \sum_J (-1)^t (1-m^{-(k_1+1)})\ldots(1-m^{-(k_t+1)})e_J e_{k_1+\ldots+k_t-1}[X]$$

where d is the degree of the finite covering $\tilde{E} \to E$ and J runs through all the subsets of I.

Proof. Using Proposition 2-1, (ii) we compute

$$\begin{aligned}\tilde{e}_I[\tilde{E}] &= \tilde{e}_{i_1}\ldots\tilde{e}_{i_r}[\tilde{E}]\\ &= m^2\bar{r}^*\{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})e^{i_1}\}\ldots\\ &\qquad m^2\bar{r}^*\{\pi^*(e_{i_r}) - (1-m^{-(i_r+1)})e^{i_r}\}[\tilde{E}]\\ &= dm^{2r}\{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})e^{i_1}\}\ldots\\ &\qquad \{\pi^*(e_{i_r}) - (1-m^{-(i_r+1)})e^{i_r}\}[E]\\ &= dm^{2r} \sum_J (-1)^t (1-m^{-(k_1+1)})\ldots(1-m^{-(k_t+1)})e_J e_{k_1+\ldots+k_t-1}[X].\end{aligned}$$

Here the last equality follows from the fact that the Gysin homomorphism $\pi_*\colon H^{2(n+1)}(E;\mathbb{Z}) \to H^{2n}(X;\mathbb{Z})$ is an isomorphism.

Proposition 2-3. For any non-negative integer n, there exists no non-trivial linear relation between the characteristic numbers of surface bundles whose base spaces are iterated surface bundles of dimension $2n$.

Proof. The assertion is clear for the case $n = 0$. We use the induction on n. Thus we assume that the assertion holds for n and prove it for $n+1$. Suppose that some linear relation $\sum_I a_I e_I = 0$ holds for $n+1$. Let us recall here that we can make the operations of our m-constructions for <u>any</u> m. Then in view of the form of the formula of Proposition 2-2, it is easy to deduce from the induction assumption that all the a_I must vanish. This completes the proof.

Next for each non-negative integer i and a partition $I = \{i_1,\dots,i_r\}$ of $n+1-i$ (n is the half of $\dim X$), we consider the number

$$\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}] = \tilde{e}^i\tilde{\pi}^*(\tilde{e}_{i_1}\dots\tilde{e}_{i_r})[\tilde{D}].$$

For each subset $J = \{j_1,\dots,j_s\}$ of I, we write $J^c = \{k_1,\dots,k_t\}$ $(s+t=r)$ as before.

Proposition 2-4. The number $\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$ is given by

$$\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$$
$$= dm^{2r+1-i}\sum_J (-1)^t(1-m^{-(k_1+1)})\dots(1-m^{-(k_t+1)})e_Je_{i+k_1+\dots+k_t-1}[X]$$

where J runs through all the subsets of I.

Proof. First we recall the fact that the restriction of the cohomology class $q^*(e)$ to D is equal to that of the cohomology class ν. Also it is easy to see that the degree of the natural map $\tilde{D}\to D$ is equal to dm. Using these facts together with Proposition 2-1, we compute

$$\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$$
$$= \{r^*(q^*(e) - \frac{m-1}{m}\nu)\}^i\ m^2r^*(\pi')^*\{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})e^{i_1}\}\dots$$
$$m^2r^*(\pi')^*\{\pi^*(e_{i_r}) - (1-m^{-(i_r+1)})e^{i_r}\}[\tilde{D}]$$
$$= dm^{2r+1-i}\ \nu^i(\pi')^*\{\pi^*(e_{i_1}) - (1-m^{-(i_1+1)})e^{i_1}\}\dots$$

$$(\pi')^*\{\pi^*(e_{i_r}) - (1 - m^{-(i_r+1)})e^{i_r}\}[D]$$

$$= dm^{2r+1-i} \sum_J (-1)^t(1 - m^{-(k_1+1)})\dots(1 - m^{-(k_t+1)})e_J e_{i+k_1+\dots+k_t-1}[X]$$

This completes the proof.

Proposition 2-5. For any natural number n, there exists no non-trivial linear relation between the numbers $\tilde{e}^i\tilde{\pi}^*(\tilde{e}_I)[\tilde{D}]$ where $\tilde{D}$ runs through all the ramification loci of m-constructions on surface bundles $\pi: E \to X$ with the base space X being iterated surface bundles of dimension $2n$.

Proof. This follows from Proposition 2-3 and Proposition 2-4 by a similar argument as that of the proof of Proposition 2-3.

3. Main theorem

The following is the main result of this paper.

Theorem 3-1. The homomorphism

$$\phi: \mathbb{Q}[e_1,\dots,e_{g-2},s_1,\dots,s_p] \longrightarrow H^*(M^p_{g,q};\mathbb{Q})$$

is injective up to degree $\frac{1}{3}g$ for all g, p and q.

To prove this we first recall a fundamental result of Harer [H] which states that the homology group $H_k(M^p_{g,q};\mathbb{Q})$ is independent of g and q provided $3k \leq g$. In terms of the cohomology group, for a fixed p the groups $H^*(M^p_{g,q};\mathbb{Q})$ are all isomorphic each other in the above range. Moreover it is easy to see that under these isomorphisms, our characteristic classes are preserved.

Next let $g_1,\dots,g_p$ be integers greater than one and let

x_j be the base point of $\Sigma_{g_j,1}$ $(j=1,\dots,p)$. We write $s^{(j)} \in H^2(M^1_{g_j,1};\mathbb{Z})$ for the s-class defined by the base point x_j (see §1). Now let g be a natural number such that $\Sigma g_j \leqq g$ and let $y_1,\dots,y_p$ be p fixed points on $\mathrm{Int}\,\Sigma_{g,1}$. We have the characteristic classes $s_j \in H^2(M^p_{g,1};\mathbb{Z})$ $(j=1,\dots,p)$ which are defined by the fixed points $y_1,\dots,y_p$. Choose any embedding of the disjoint union $\coprod_{j=1}^{p} \Sigma_{g_j,1}$ of compact surfaces $\Sigma_{g_j,1}$ with boundaries into $\Sigma_{g,1}$ such that the point x_j goes to y_j $(j=1,\dots,p)$. This induces a homomorphism

$$\iota\colon M^1_{g_1,1} \times \dots \times M^1_{g_p,1} \longrightarrow M^p_{g,1}.$$

Proposition 3-2. (i) $\iota^*(e_i) = \sum_{j=1}^{p} p_j^*(e_i)$ for all $i \geqq 1$, where $p_j\colon M^1_{g_1,1} \times \dots \times M^1_{g_p,1} \to M^1_{g_j,1}$ is the projection to the j-th factor.

(ii) $\iota^*(s_j) = p_j^*(s^{(j)})$ for all $j=1,\dots,p$.

Proof. The first statement is proved in [M 1], Proposition 3-4. The second one is clear.

Proof of Theorem 3-1. First recall from §1 that the total space E of any surface bundle $\pi\colon E \to X$ serves as the base space of the associated pull back surface bundle $\pi'\colon E^* \to E$ which has a canonical cross section. Moreover the s-class of π' is equal to the Euler class $e \in H^2(E;\mathbb{Z})$ of π. Then in view of Harer's result mentioned above, the required assertion for the case $p = 1$ follows from Proposition 2-5. The general cases follow from this by an easy argument using Proposition 3-2. This completes the proof.

Remark 3-3. According to a recent result of Harer-Zagier [HZ], the homomorphism ϕ is far from being surjective. However it seems to be still reasonable to conjecture that our characteristic classes exhaust all the "stable" characteristic classes of surface bundles, namely the natural homomorphism

$$\mathbb{Q}[e_1, e_2, \ldots; s_1, \ldots, s_p] \longrightarrow \lim_{g\to\infty} H^*(M^p_{g,q}; \mathbb{Q})$$

would be an isomorphism. Theorem 3-1 shows that it is in fact injective.

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